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A Lax pair for a lattice modified KdV equation, reductions to q-Painlevé equations and associated Lax pairs

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Abstract
We present a new, nonautonomous Lax pair for a lattice nonautomous modified Korteweg–deVries equation and show that it can be consistently extended multi-dimensionally, a property commonly referred to as being consistent around a cube. This nonautonomous equation is reduced to a series of q-discrete Painlevé equations, and Lax pairs for the reduced equations are found. A 2×2 Lax pair is given for a $q\mathrm{P}_{\mathrm{III}}$ with multiple parameters and, also, for versions of $q\mathrm{P}_{\mathrm{II}}$ and $q\mathrm{P}_{\mathrm{V}}$, all for the first time.

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1. Introduction

Nonlinear evolution equations occur frequently in physical modelling and applied mathematics. Nonlinear integrable lattices provide a natural discrete extension of classically integrable systems. More recently, there has been great interest in nonlinear ordinary difference equations. We consider reductions from lattice equations to ordinary difference equations which constitute a natural link between the two classes of equations. Our main perspective will lie in the construction of Lax pairs for difference equations.

Most studies [1–6] of reductions of lattice equations focus on equations in which all parameters are independent of lattice variables. For example, the lattice modified Korteweg–de Vries equation [2],

$$\text{LMKdV:} \quad x_{l+1,m+1} = x_{l,m} \frac{(p x_{l+1,m} - q x_{l,m+1})}{(p x_{l,m+1} - q x_{l+1,m})},$$

contains lattice parameters p, q which are considered to be independent of the lattice variables l, m. In [7], a new type of reduction from the lattice equations to ordinary difference equations was introduced by starting with non-autonomous lattice equations. In this approach, the lattice parameters p, q were considered to be functions of l, m, under the condition that the lattice equation satisfied the singularity confinement property. Such non-autonomous forms of well-known lattice equations were then shown to reduce to q-discrete Painlevé equations, including $q\mathrm{P}_{\mathrm{II}}$, $q\mathrm{P}_{\mathrm{III}}$ and $q\mathrm{P}_{\mathrm{V}}$.

Such q-discrete Painlevé equations are of fundamental interest in the theory of integrable systems and random matrix theory. We note that the full generic form of $q\mathrm{P}_{\mathrm{III}}$ was first obtained in [8]. Its natural generalization is a q-discrete sixth Painlevé equation ($q\mathrm{P}_{\mathrm{VI}}$) first obtained in [9]. The integrability of such equations lies in the fact that they can be solved through an associated linear problem called a Lax pair. For $q\mathrm{P}_{\mathrm{III}}$ the Lax pair was obtained in [10], with a notable feature that the linear problem is a matrix problem involving matrices of size 4×4. On the other hand, the Lax pairs of lattice equations, such as the LMKdV [1, 2], and many discrete Painlevé equations, such as $q\mathrm{P}_{\mathrm{VI}}$ are typically matrix problems of size 2×2. In [15], a 2×2 Lax pair was given for a special case of $q\mathrm{P}_{\mathrm{III}}$.

In this paper, we present two types of results. First, we show that an extension of the reduction method given in [7] is possible and, by using the extension, deduce a sequence of discrete Painlevé equations as reductions of lattice equations. Second, we give a Lax pair of the non-autonomous LMKdV and show that it gives rise to 2×2 matrix Lax pairs under the reductions to q-Painlevé equations. In obtaining the latter, a key observation was needed that arises from the multi-dimensional embedding of lattice equations in a self-consistent way in three directions. The resulting theory [11–14] is often referred to as 'consistency around a cube'.

This paper is organized as follows. In section 2, we recall the Lax pair of LMKdV and generalize it to provide a non-autonomous Lax pair for the non-autonomous version of LMKdV. We also show that this Lax pair and the generalized LMKdV form a multi-dimensional system that satisfies the self-consistency property. In section 3, we consider the reductions of the non-autonomous LMKdV to ordinary difference equations and provide extensions of previously considered reductions. In section 4, we show that 2×2 Lax pairs for the reductions can be found by applying the idea of self-consistency and reductions to the Lax pair of the LMKdV. We end this paper with a conclusion where we also point out some open problems.

2. Lax pair and self-consistency of the non-autonomous LMKdV

While a linear problem, or Lax pair, associated with the LMKdV has been known for a long time [1, 2], it appears that linear problems associated with the non-autonomous version of the LMKdV have not been written down before. We provide an explicit Lax pair for the non-autonomous version of the LMKdV in subsection 2.1.

Furthermore, while the theory of multi-dimensional extensions of lattice equations has been explored fairly widely, the theory has not been applied explicitly to non-autonomous lattice equations. We provide such an application to the non-autonomous LMKdV in subsection 2.2.

2.1. Lax pair of the non-autonomous LMKdV

A Lax pair for the LMKdV is a linear problem of the form

$$v(l+1, m) = L(l, m)v(l, m), \qquad v(l, m+1) = M(l, m)v(l, m), \tag{2.1}$$

whose compatibility condition, namely, $L(l, m+1)M(l, m) = M(l+1, m)L(l, m)$, is the LMKdV.

Hereafter we adopt the notation $\overline{v} = v(l+1, m)$ and $\hat{v} = v(l, m+1)$. (We have used l in place of the more traditional n here because it is notationally more appropriate that the L matrix should create a shift in l and M in m. Later, in section 3, we will see that a third Lax matrix, N, arises whose associated shifts will be in n.) Now set

$$L = \begin{pmatrix} \overline{x}/x & -\lambda/(\nu x) \\ -\lambda\overline{x}/\nu & 1 \end{pmatrix}, \tag{2.2a}$$

$$M = \begin{pmatrix} \hat{x}/x & -\mu/(\nu x) \\ -\mu\hat{x}/\nu & 1 \end{pmatrix}, \tag{2.2b}$$

where ν is a spectral variable, μ is a function of m alone, and λ is a function of l alone.

Compatibility occurs when $\widehat{L}M = \overline{M}L$. In this equation, it is straightforward to check that the diagonal entries yield identities and the off-diagonal entries each contain the lattice mKdV equation in the following way. The top-right entry yields

$$\frac{\mu\hat{\overline{x}}}{\hat{x}x} + \frac{\lambda}{\hat{x}} = \frac{\lambda\hat{\overline{x}}}{\overline{x}x} + \frac{\mu}{\overline{x}} \quad \Rightarrow \quad \hat{\overline{x}}(\mu\overline{x} - \lambda\hat{x}) = x(\mu\hat{x} - \lambda\overline{x}).$$

Similarly, the bottom left entry yields the same equation. Thus we arrive at the following form of the LMKdV equation,

$$\hat{\overline{x}} = x\frac{\overline{x} - r\hat{x}}{\hat{x} - r\overline{x}}, \tag{2.3}$$

where we have introduced

$$r(l, m) = \frac{\mu(m)}{\lambda(l)}. \tag{2.4}$$

This form of the LMKdV equation is identical to the one used in [7], except for a factor of (-1) which is inconsequential. Indeed, we can achieve the equation used there exactly if we premultiply each of L and M by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We use the slightly different form here simply because it allows for a more symmetric Lax pair. In [7] it is noted that r must separate as in (2.4) because it has to satisfy

$$\hat{\overline{r}}r = \hat{r}\overline{r} \tag{2.5}$$

for the singularity confinement property to be satisfied. We note that integrability conditions for lattice equations have also been studied recently in [22].

2.2. Consistency around a cube

In this subsection, we show that the Lax pair, (2.2), given in the previous subsection, is multi-dimensionally consistent with the lattice equation LMKdV.

In this point of view, the lattice variables l, m provide a two-dimensional slice of a three-dimensional space in which the third direction, coordinatized by n say, can be thought of as providing the spectral direction for the Lax pair. The shifts $l \mapsto l+1, m \mapsto m+1, n \mapsto n+1$ describe a fundamental cube in this multi-dimensional space. The term 'consistent around a cube' arises from the fact that the iteration of the map on any face of the fundamental cube provides a corner value that is the same as that provided by iteration on an intersecting face.

Define $\tilde{x} = x(l, m, n+1)$, such that $\tilde{x} = u/t$, where u and t are the components of the eigenfunction $v(l, m, n)$, i.e.,

$$v = \begin{pmatrix} t \\ u \end{pmatrix}$$

where v satisfies the linear system (2.1).

Since $\overline{v} = Lv$, so

$$\begin{aligned} \overline{\tilde{x}} &= \overline{u}/\overline{t} \\ &= \frac{u - \lambda \overline{x} t/\nu}{\overline{x} t/x - \lambda u/(\nu x)} \\ \overline{\tilde{x}} &= x \frac{\tilde{x} - \rho \overline{x}}{\overline{x} - \rho \tilde{x}} \end{aligned} \tag{2.6}$$

where we have allowed ν to depend on n and replaced λ/ν by $\rho(l, n)$. Since M takes on the same form as L, we can clearly find an equivalent expression in the m and n directions. And, because (2.6) is the LMKdV equation again, we conclude that the Lax pair is multi-dimensionally consistent with the LMKdV equation.

Essentially we have done the reverse of the usual operation. Ordinarily one begins with a system that is consistent around a cube and then constructs its Lax pair (see [14] or [11, 12]). However, here we began with the Lax pair and showed that it is multi-dimensionally consistent with the LMKdV equation.

3. Reductions to ordinary difference equations

In this section, we consider reductions from the partial difference equation (2.3) to a sequence of ordinary difference equations. These include $q\mathrm{P}_{\mathrm{II}}$, a three-parameter version of $q\mathrm{P}_{\mathrm{III}}$, a special case of $q\mathrm{P}_{\mathrm{V}}$, and, moreover, some higher-order difference equations. We present the results in a series of subsections.

Let $\hat{x} = f(\mathbf{x})$ where $\mathbf{x}$ represents x and its iterates, $\overline{x}, \overline{\overline{x}}, \ldots$. Thus, (2.3) becomes

$$\overline{f(\mathbf{x})} = x \frac{\overline{x} - r f(\mathbf{x})}{f(\mathbf{x}) - r\overline{x}}. \tag{3.1}$$

For $f(\mathbf{x})$ to be acceptable, it must produce the same reduced equation when we begin with the mKdV equation iterated up once in m. That is

$$\hat{\overline{\hat{x}}} = \hat{x} \frac{\hat{\overline{x}} - \hat{r}\hat{\hat{x}}}{\hat{\hat{x}} - \hat{r}\hat{\overline{x}}} \tag{3.2}$$

must lead to the same reduction, with possible conditions placed on r. We note that $\hat{x} = f(\mathbf{x})$ so $\hat{\hat{x}} = f(\hat{\mathbf{x}}) = f(f(\mathbf{x}))$ and so (3.2) is equivalent to

$$f(f(\overline{\mathbf{x}})) = f(\mathbf{x}) \frac{f(\overline{\mathbf{x}}) - \hat{r} f(f(\mathbf{x}))}{f(f(\mathbf{x})) - \hat{r} f(\overline{\mathbf{x}})}. \tag{3.3}$$

We will now consider some specific possible cases of $f(\mathbf{x})$.

3.1. $f(\mathbf{x}) = \overline{x}^{\alpha}$

The first reduction we consider is $f(\mathbf{x}) = \overline{x}^{\alpha}$ in (3.1) so that

$$\overline{\overline{x}}^{\alpha} = x \frac{\overline{x}^{1-\alpha} - r}{1 - r\overline{x}^{1-\alpha}} \tag{3.4}$$

and we use the same $f(\mathbf{x})$ in (3.2) to get

$$\overline{\overline{x}}^{\alpha} = x\left[\frac{\overline{x}^{\alpha(1-\alpha)} - \underline{\hat{r}}}{1-\underline{\hat{r}}\,\overline{x}^{\alpha(1-\alpha)}}\right]^{1/\alpha}. \tag{3.5}$$

The two expressions for $\overline{\overline{x}}^{\alpha}$, (3.4) and (3.5), agree if $\alpha = 1$ but this leads to a linear equation. Another solution is $\alpha = -1$ and $\hat{r} = \overline{r}$. The latter condition on r dictates through (2.5) that $r = \beta\gamma^{l}$, where both β and γ are constant, so that the final form of the reduced equation is

$$\overline{\overline{x}}x = \frac{\beta\gamma^{l}\overline{x}^{2} - 1}{\beta\gamma^{l} - \overline{x}^{2}} \tag{3.6}$$

which is a special case of a q-discrete Painlevé III equation ($q\mathrm{P}_{\mathrm{III}}$) found in [8].

Equation (3.6) was already obtained in [7] as a reduction of the lattice sine-Gordon equation (LSG). The advantage of the reduction presented above is that it comes with a Lax pair (see section 4.2). As a point of interest we mention that the LMKdV can be transformed to the LSG by using $\hat{x} \to 1/\hat{x}$.

3.2. $f(\mathbf{x}) = \overline{\overline{x}}^{\alpha}$

Now consider $f(\mathbf{x}) = \overline{\overline{x}}^{\alpha}$ the same analysis as above shows that, again, $\alpha = 1$ or $\alpha = -1$ will lead to valid reductions. When $\alpha = 1$, we must set $\log r = al + b + c(-1)^{l}$ and, after introducing $y = \overline{\overline{x}}/x$, we are left with

$$\overline{y}\underline{y} = \frac{1-ry}{y(y-r)}. \tag{3.7}$$

The same equation as (3.7) was found in [7] where the equation was identified as either $q\mathrm{P}_{\mathrm{II}}$ or $q\mathrm{P}_{\mathrm{III}}$, depending on whether $c = 0$.

Now take the case when $\alpha = -1$, this time (3.1) becomes

$$\overline{\overline{x}}x = \frac{\overline{\overline{\overline{x}x}x}x}{\overline{\overline{x}x}} = \frac{r\overline{\overline{x}x} - 1}{r - \overline{\overline{x}x}}$$

whereupon setting $y = \overline{\overline{x}x}$ we find

$$\overline{y}\underline{y} = y\frac{ry-1}{r-y}. \tag{3.8}$$

To find the required form of the parameter functions we must compare this to (3.2) with the same y substituted

$$\overline{\overline{\overline{y}}y} = \overline{\overline{y}}\frac{1-\hat{r}\overline{\overline{y}}}{\overline{\overline{y}}-\hat{r}}.$$

Clearly, the equivalence between these two mappings is satisfied by taking r as for the case when $\alpha = +1$. Equation (3.8) is actually equivalent to (3.7) and was also derived in [7] but from the lattice sine-Gordon equation rather than the l mKdV.

3.3. $f(\mathbf{x}) = \overline{\overline{\overline{x}}}$

We let $f = \overline{\overline{\overline{x}}}$ and, on substituting $w = \overline{\overline{\overline{x}}}/x$, we have

$$\overline{w}\underline{w} = \frac{1-rw}{w-r}. \tag{3.9}$$

Here $\log r = al + b + c\xi^{l} + d\xi^{2l}$, $a, \ldots, d =$ constants and $\xi^{3} = 1$. This equation was shown to be a $q\mathrm{P}_{\mathrm{II}}$ when $c = d = 0$ [19] or a $q\mathrm{P}_{\mathrm{V}}$ in the general case [20].

3.4. $f(\mathbf{x}) = 1/\overline{\overline{\overline{x}}}$

Specifying $f = 1/\overline{\overline{\overline{x}}}$ leads to what appears to be an irreducible, fourth order, integrable difference equation, namely

$$\overline{\overline{x}}\underline{\underline{x}} = \frac{\overline{x}\underline{x} - r}{1 - r\overline{x}\underline{x}} \tag{3.10}$$

where again $\log r = al + b + cj^l + dj^{2l}$ and $j^3 = 1$.

3.5. $f(\mathbf{x}) = \overline{\overline{\overline{\overline{x}}}}$

The reductions of orders higher than third all lead to equations that are not reducible to a second-order form. The next reduction to consider is $f = x_{l+4}$ which becomes

$$\overline{\overline{y}}\,\overline{y}y\underline{y}\underline{\underline{y}} = \frac{1 - r\overline{y}y\underline{y}}{\overline{y}y\underline{y} - r} \tag{3.11}$$

with $y = \overline{\overline{\overline{x}}}/\overline{x}$, and $\log r = a(-1)^l + b + cl + d\cos\left(\frac{l\pi}{2}\right) + e\sin\left(\frac{l\pi}{2}\right)$, $a, \ldots, e =$ constants.

3.6. $f(\mathbf{x}) = 1/\overline{\overline{\overline{\overline{x}}}}$

Using $f = 1/x_{l+4}$ allows the reduction of the lmKdV to

$$\overline{\overline{y}}y\underline{\underline{y}} = \overline{y}\underline{y}\frac{r\overline{y}\underline{y} - y}{ry - \overline{y}\underline{y}} \tag{3.12}$$

where r is the same as in the previous example but $y = \overline{\overline{\overline{x}}}\,\overline{\overline{x}}$.

Arbitrarily high-order equations can be generated in this manner.

4. Lax pairs for the reduced equations

In this section, we deduce Lax pairs for the q-discrete Painlevé equations derived in the previous section by applying the observations obtained from the multi-dimensional self-consistency of the LMKdV system and its Lax pair. Since the reductions leading to $q\mathrm{P}_{\mathrm{II}}$, $q\mathrm{P}_{\mathrm{III}}$ and $q\mathrm{P}_{\mathrm{V}}$ differ, we give the details of each separately in three subsections.

So far the linear system is given by (2.1) but now we wish to include the third direction, n, that we introduced in section 2.2. The variable n will come into L via ν which plays the role of the spectral variable, i.e., we allow $\nu = \nu(n)$. This gives rise to the reduced equations through compatibility between the l and n directions. We write $L\big(l, m, \nu(n)\big) = L(l, m, n)$, $M\big(l, m, \nu(n)\big) = M(l, m, n)$ and introduce a matrix $N\big(l, m, \nu(n)\big) = N(l, m, n)$ such that

$$v(l+1, m, n) = L(l, m, \nu(n))v(l, m, n) \tag{4.1a}$$

$$v(l, m, n+1) = N(l, m, \nu(n))v(l, m, n), \tag{4.1b}$$

where L in the first equation is the same as in equation (2.2a). We indicate a shift in n by $a(l, m, \nu(n+1)) =: a(l, m, n+1) =: \tilde{a}$. Now the compatibility condition of the above two equations is $\tilde{L}N = \overline{N}L$.

We label the components of v, as in section 2.2, by

$$v = \begin{pmatrix} t \\ u \end{pmatrix} \tag{4.2}$$

and let $\tilde{x} = u/t$. We are now in a position to find the form of N through the reduction.

4.1. Lax pair for $q\mathrm{P}_{\mathrm{II}}$

First consider the reduction $\hat{x} = \overline{\overline{x}}$ which reduces the LMKdV equation to $q\mathrm{P}_{\mathrm{II}}$. Recall that $\overline{v} = Lv$ and $\hat{v} = Mv$ so on the one hand,

$$\underset{\sim}{\hat{v}} = \begin{pmatrix} \underset{\sim}{\hat{t}} \\ \underset{\sim}{\hat{u}} \end{pmatrix} = \underset{\sim}{\hat{t}} \begin{pmatrix} 1 \\ \hat{x} \end{pmatrix} = \underset{\sim}{\hat{t}} \begin{pmatrix} 1 \\ \overline{\overline{x}} \end{pmatrix}$$

and on the other hand,

$$\underset{\sim}{\overline{\overline{v}}} = \underset{\sim}{\overline{\overline{t}}} \begin{pmatrix} 1 \\ \overline{\overline{x}} \end{pmatrix} = (\underset{\sim}{\overline{\overline{t}}}/\underset{\sim}{\hat{t}})\underset{\sim}{\hat{v}}.$$

But $\underset{\sim}{\hat{v}} = (\underset{\sim}{\widehat{N}})^{-1}Mv$ and $\underset{\sim}{\overline{\overline{v}}} = (\underset{\sim}{\overline{\overline{N}}})^{-1}\overline{L}Lv$. Thus

$$\underset{\sim}{\overline{\overline{t}}}\,\underset{\sim}{\overline{\overline{N}}} = \overline{L}LM^{-1}\underset{\sim}{\hat{t}}\,\underset{\sim}{\widehat{N}}. \tag{4.3}$$

We now use (4.3) as a guide and try a general N that has the same form as $\overline{L}LM^{-1}$, where by the same form we mean that it contains the same powers of ν. From equations (2.2), we find

$$\overline{L}LM^{-1} = \frac{1}{\nu^2-\mu^2}\begin{pmatrix} \nu^2 + \lambda\overline{\lambda}\frac{x}{\overline{\overline{x}}} - \mu\lambda\frac{\overline{\overline{x}}}{\overline{x}} - \mu\overline{\lambda}\frac{x}{\overline{x}} & \nu\left(\frac{\mu}{x} - \lambda\frac{\overline{\overline{x}}}{\overline{x}x} - \frac{\overline{\lambda}}{\overline{\overline{x}}}\right) + \frac{1}{\nu}\mu\overline{\lambda}\lambda/\overline{\overline{x}} \\ \nu\left(\mu x - \overline{\lambda}\overline{x} - \lambda\frac{\overline{x}x}{\overline{\overline{x}}}\right) + \frac{1}{\nu}\mu\overline{\lambda}\lambda\overline{\overline{x}} & \nu^2 + \overline{\lambda}\lambda\frac{\overline{\overline{x}}}{x} - \mu\overline{\lambda}\frac{\overline{x}}{x} - \mu\lambda\frac{\overline{x}}{\overline{\overline{x}}} \end{pmatrix}.$$

Since the prefactor cancels in the compatibility condition (4.5), we take N to be

$$N = \begin{pmatrix} \nu^2 a_2 + a_0 & \nu b_1 + b_0/\nu \\ \nu c_1 + c_0/\nu & \nu^2 d_2 + d_0 \end{pmatrix} \tag{4.4}$$

where the coefficients a_i, b_i, c_i and d_i are functions of l only. Note that ν is the spectral parameter and needs to be related to the spectral variable n. We will assume $\nu(n) = q^n$ and continue to use ν as the spectral variable in L and N. However, there is an important observation to be made after reduction: x should be independent of ν. (This is the property that all discrete Painlevé equations possess: their solutions are independent of the spectral variable that occurs in their respective Lax pairs.) Recall that shifts in n are denoted by $\tilde{}$. Hence we take $\tilde{x} = x$ in the following.

Now the coefficients of the various powers of ν in N are determined by the compatibility condition

$$\tilde{L}N = \overline{N}L \tag{4.5}$$

which is the compatibility condition of the system (4.1). Going through the calculations in detail would be somewhat tedious so only an outline will be given here. The compatibility condition gives a total of ten equations, three in each of the diagonal entries and two in the off-diagonal entries. The equations in the diagonal entries at order ν^2 and ν^{-2} are solved in a straightforward manner, yielding

$$\begin{aligned} a_2 &= \text{constant} \\ d_2 &= \text{constant} \\ b_0 &= \frac{\gamma}{x\sigma} \\ c_0 &= \frac{\gamma x}{\sigma} \end{aligned} \tag{4.6}$$

where γ is a constant and $\sigma = q^l$. The remaining six equations read as follows:

$$\overline{a_0} - a_0 = \lambda(x\overline{b_1} - c_1/q\overline{x}) \tag{4.7a}$$

$$\overline{d_0} - d_0 = \lambda(\overline{c_1}/x - \overline{x}b_1/q) \tag{4.7b}$$

$$\overline{b_1}x - b_1\overline{x} = \lambda(a_2 - d_2/q) \tag{4.7c}$$

$$\overline{c_1}/x - c_1/\overline{x} = \lambda(d_2 - a_2/q) \tag{4.7d}$$

$$\overline{a_0} - d_0/q = \frac{\gamma}{\lambda\sigma}\left(\frac{x}{q\overline{x}} - \frac{\overline{x}}{x}\right) \tag{4.7e}$$

$$\overline{d_0} - a_0/q = \frac{\gamma}{\lambda\sigma}\left(\frac{\overline{x}}{qx} - \frac{x}{\overline{x}}\right). \tag{4.7f}$$

To solve these, use (4.7c) to replace $\overline{b_1}x$ in (4.7a), and the resulting expression to replace $\overline{a_0}$ in (4.7e). Now do the same with (4.7d) and (4.7b) in (4.7f), then solve these two equations for a_0 and d_0 to find

$$a_0 = -\frac{\gamma\overline{x}}{\lambda\sigma x} - \lambda b_1\overline{x} - \lambda^2 a_2 \tag{4.8a}$$

$$d_0 = -\frac{\gamma x}{\lambda\sigma\overline{x}} - \frac{\lambda c_1}{\overline{x} - \lambda^2 d_2}. \tag{4.8b}$$

One can now use (4.8a) and (4.8b) in equations (4.7a) and (4.7b), replace $\overline{b_1}$ and $\overline{c_1}$ via (4.7c) and (4.7d), then solve the remainder for b_1 and c_1. All this reduces to

$$a_0 = \frac{a_2\lambda\overline{\lambda}x}{\overline{\overline{x}}} - \frac{\gamma\overline{x}}{\sigma}\left(\frac{1}{\overline{\lambda\overline{x}}} + \frac{1}{\lambda x}\right) \tag{4.9a}$$

$$d_0 = \frac{d_2\lambda\overline{\lambda\overline{x}}}{x} - \frac{\gamma}{\sigma\overline{x}}\left(\frac{\overline{\overline{x}}}{\overline{\overline{\lambda}}} + \frac{x}{\lambda}\right) \tag{4.9b}$$

$$b_1 = \frac{\gamma}{\lambda\overline{\lambda}\sigma\overline{\overline{x}}} - \frac{a_2}{\overline{x}}\left(\lambda + \overline{\lambda}\frac{x}{\overline{\overline{x}}}\right) \tag{4.9c}$$

$$c_1 = \frac{\gamma\overline{\overline{x}}}{\lambda\overline{\lambda}\sigma} - d_2\overline{x}\left(\lambda + \overline{\lambda}\frac{\overline{\overline{x}}}{x}\right). \tag{4.9d}$$

Finally, these calculated values should be substituted back into equations (4.7a)–(4.7f). On doing this and making the substitution $y = \overline{\overline{x}}/\overline{x}$, we find that one of the two following forms of $q\mathrm{P_{II}}$ arises in each case

$$\underline{y}\overline{y} = \frac{1}{y}\frac{\gamma\lambda y - a_2\lambda\overline{\lambda\overline{\lambda}}^2 q\sigma}{q\gamma\overline{\overline{\lambda}} - d_2\lambda^2\overline{\lambda\overline{\lambda}}\sigma y} \tag{4.10}$$

$$\underline{y}\overline{y} = \frac{1}{y}\frac{\gamma\overline{\overline{\lambda}}qy - a_2\lambda^2\overline{\lambda\overline{\lambda}}\sigma}{\gamma\lambda - d_2\lambda\overline{\lambda\overline{\lambda}}^2 q\sigma y}. \tag{4.11}$$

Equations (4.10) and (4.11) can be reconciled by setting $\lambda = AB^{(-1)^l}q^{l/2}$, with constants A and B, which gives the same form of $q\mathrm{P_{II}}$ that was given earlier (and found in [7]) by a reduction from the LMKdV equation. Hence we have calculated a Lax pair for $q\mathrm{P_{II}}$, which explicitly takes the form

$$v(l+1,n) = L(l,\nu(n))v(l,n) \tag{4.12a}$$

$$v(l,n+1) = N(l,\nu(n))v(l,n) \tag{4.12b}$$

where

$$L = \begin{pmatrix} \frac{\overline{x}}{x} & -\frac{\lambda}{\nu x} \\ \frac{-\lambda\overline{x}}{\nu} & 1 \end{pmatrix}, \tag{4.13}$$

$$N = \begin{pmatrix} a_2\nu^2 + \frac{a_2\lambda\overline{\lambda}x}{\overline{\overline{x}}} - \frac{\gamma\overline{x}}{\sigma}\left(\frac{1}{\overline{\overline{\lambda x}}} + \frac{1}{\lambda x}\right) & \nu\left[\frac{\gamma}{\lambda\overline{\lambda}\sigma\overline{\overline{x}}} - \frac{a_2}{\overline{\overline{x}}}\left(\lambda + \overline{\lambda}\frac{x}{\overline{\overline{\overline{x}}}}\right)\right] + \frac{\gamma}{\nu\sigma x} \\ \nu\left[\frac{\gamma\overline{\overline{x}}}{\lambda\overline{\lambda}\sigma} - d_2\overline{x}\left(\lambda + \overline{\lambda}\frac{\overline{\overline{x}}}{x}\right)\right] + \frac{\gamma x}{\nu\sigma} & d_2\nu^2 + \frac{d_2\lambda\overline{\lambda}\overline{\overline{x}}}{x} - \frac{\gamma}{\sigma\overline{x}}\left(\frac{\overline{\overline{x}}}{\overline{\overline{\lambda}}} + \frac{x}{\lambda}\right) \end{pmatrix}. \tag{4.14}$$

4.2. *Lax pair for* $q\mathrm{P}_{\mathrm{III}}$

The next reduction to be considered is that taking LMKdV to $q\mathrm{P}_{\mathrm{III}}$, i.e., $\hat{x} = 1/\overline{x}$. The reciprocal in the latter reduction introduces a difference in the method used to find the corresponding Lax pair. We now have

$$\underset{\sim}{\hat{v}} = \underset{\sim}{\hat{t}}\begin{pmatrix} 1 \\ \hat{x} \end{pmatrix} = \underset{\sim}{\hat{t}}\begin{pmatrix} 1 \\ 1/\bar{x} \end{pmatrix} = \underset{\sim}{\hat{t}}/\bar{x}\begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}$$

and

$$\overline{v} = \underset{\sim}{\overline{t}}\begin{pmatrix} 1 \\ \overline{x} \end{pmatrix} = \frac{\bar{x}\underset{\sim}{\overline{t}}}{\underset{\sim}{\hat{t}}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underset{\sim}{\hat{v}}\,.$$

So this time the suggested form of N is the same as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)LM^{-1}$ and, as such, we choose

$$N = \begin{pmatrix} a/\nu & b_0 + b_2/\nu^2 \\ c_0 + c_2/\nu^2 & d/\nu \end{pmatrix}. \tag{4.15}$$

In this case the compatibility condition contains eight equations that are solved in a similar way to before, and these lead to

$$a = -\lambda\beta x\overline{x} - \frac{\alpha\overline{x}}{\lambda\sigma x} \tag{4.16a}$$

$$b_0 = \beta x \tag{4.16b}$$

$$b_2 = \frac{\alpha}{\sigma x} \tag{4.16c}$$

$$c_0 = \frac{\gamma}{x} \tag{4.16d}$$

$$c_2 = \frac{\alpha x}{\sigma} \tag{4.16e}$$

$$d = -\frac{\lambda\gamma}{x\overline{x}} - \frac{\alpha x}{\lambda\sigma\overline{x}} \tag{4.16f}$$

where α, β and γ are all constants, $\lambda = \delta q^{-l}$ where $\delta =$ constant and $\sigma = q^l$.

The final form of the $q\mathrm{P}_{\mathrm{III}}$ equation obtained from the compatibility conditions is

$$x\overline{\overline{x}} = \frac{\mu_1 q^l\overline{x}^2 + \mu_2}{\mu_3 q^l + \overline{x}^2} \tag{4.17}$$

which is a non-autonomous equation with three free parameters μ_i. The Lax pair is of the form

$$v(l+1,n) = L(l,\nu(n))v(l,n) \tag{4.18a}$$

$$v(l,n+1) = N(l,\nu(n))v(l,n) \tag{4.18b}$$

where L as before is given by

$$L = \begin{pmatrix} \frac{\overline{x}}{x} & -\frac{\lambda}{\nu x} \\ \frac{-\lambda\overline{x}}{\nu} & 1 \end{pmatrix}, \tag{4.19}$$

and

$$N = \begin{pmatrix} -\frac{1}{\nu}\left(\lambda\beta x\overline{x} + \frac{\alpha\overline{x}}{\lambda\sigma x}\right) & \beta x + \frac{\alpha}{\nu^2\sigma x} \\ \frac{\gamma}{x} + \frac{\alpha x}{\nu^2\sigma} & -\frac{1}{\nu}\left(\frac{\lambda\gamma}{x\overline{x}} + \frac{\alpha x}{\lambda\sigma\overline{x}}\right) \end{pmatrix}. \tag{4.20}$$

4.3. Lax pair for $q\mathrm{P}_{\mathrm{V}}$

Lastly, a Lax pair for $q\mathrm{P}_{\mathrm{V}}$ (see equation (3.9)) is presented. Following the previous analysis, we begin with N of the same form as $\overline{\overline{L}}\,\overline{L}LM^{-1}$ or

$$N = \begin{pmatrix} a_1\nu^2 + a_0 + a_2/\nu^2 & b_1\nu + b_0/\nu \\ c_1\nu + c_0/\nu & d_1\nu^2 + d_0 + d_2/\nu^2 \end{pmatrix}. \tag{4.21}$$

After similar methods as we used earlier, we arrive at the following, where $\log(T_2) = A + B(-1)^l$ is a function of period two, and $\sigma = q^l$ as before.

$$\begin{aligned}
a_0 &= \frac{T_2\overline{x}}{\sigma\overline{\overline{\overline{x}}}}\left(\frac{\overline{\overline{x}}}{\overline{\overline{\lambda}}\lambda x} + \frac{\overline{\overline{x}}\,\overline{\overline{\overline{x}}}}{\lambda\overline{\lambda}x\overline{x}} + \frac{1}{\overline{\overline{\lambda}}\,\overline{\lambda}}\right) + a_1\left(\lambda\overline{\lambda}\frac{x}{\overline{\overline{\overline{x}}}} + \lambda\overline{\overline{\lambda}}\frac{x\overline{x}}{\overline{\overline{\overline{x}}}\,\overline{\overline{\overline{x}}}} + \overline{\lambda}\,\overline{\overline{\lambda}}\frac{\overline{x}}{\overline{\overline{\overline{x}}}}\right)\\
a_1 &= \text{constant}\\
a_2 &= \frac{T_2}{\sigma}\\
d_0 &= \frac{\overline{T_2}\,\overline{\overline{\overline{x}}}}{\sigma\overline{x}}\left(\frac{x}{\lambda\overline{\overline{\lambda}}\,\overline{\overline{x}}} + \frac{x\overline{x}}{\lambda\overline{\lambda}\,\overline{\overline{x}}\,\overline{\overline{\overline{x}}}} + \frac{1}{\overline{\lambda}\,\overline{\overline{\lambda}}}\right) + d_1\left(\lambda\overline{\lambda}\frac{\overline{\overline{x}}}{x} + \lambda\overline{\overline{\lambda}}\frac{\overline{\overline{x}}\,\overline{\overline{\overline{x}}}}{x\overline{x}} + \overline{\lambda}\,\overline{\overline{\lambda}}\frac{\overline{\overline{\overline{x}}}}{\overline{x}}\right)\\
d_1 &= \text{constant}\\
d_2 &= \frac{\overline{T_2}}{\sigma}\\
b_0 &= -\frac{T_2\overline{\overline{x}}}{\sigma x}\left(\frac{1}{\overline{\overline{\lambda}}\,\overline{\overline{\overline{x}}}} + \frac{1}{\overline{\lambda}\,\overline{x}} + \frac{x}{\lambda\overline{x}\,\overline{\overline{x}}}\right) - a_1\frac{\lambda\overline{\lambda}\,\overline{\overline{\lambda}}}{\overline{\overline{\overline{x}}}}\\
b_1 &= -\frac{T_2}{\lambda\overline{\lambda}\,\overline{\overline{\lambda}}\sigma\overline{\overline{\overline{x}}}} - \frac{a_1}{\overline{\overline{\overline{x}}}}\left(\lambda\frac{\overline{\overline{\overline{x}}}}{\overline{x}} + \overline{\lambda}\frac{x\overline{\overline{\overline{x}}}}{\overline{x}\,\overline{\overline{x}}} + \overline{\overline{\lambda}}\frac{x}{\overline{\overline{\overline{x}}}}\right)\\
c_0 &= -\frac{\overline{T_2}x}{\sigma\overline{\overline{x}}}\left(\frac{\overline{\overline{\overline{x}}}}{\overline{\overline{\lambda}}} + \frac{\overline{x}}{\overline{\lambda}} + \frac{\overline{x}\,\overline{\overline{x}}}{\lambda x}\right) - d_1\lambda\overline{\lambda}\,\overline{\overline{\lambda}}\,\overline{\overline{\overline{x}}}\\
c_1 &= -\frac{\overline{T_2}\,\overline{\overline{\overline{x}}}}{\lambda\overline{\lambda}\,\overline{\overline{\lambda}}\sigma} - d_1\overline{\overline{\overline{x}}}\left(\lambda\frac{\overline{x}}{\overline{\overline{\overline{x}}}} + \overline{\lambda}\frac{\overline{x}\,\overline{\overline{x}}}{x\overline{\overline{\overline{x}}}} + \overline{\overline{\lambda}}\frac{\overline{\overline{x}}}{x}\right).
\end{aligned}$$

We also find that $\log\lambda = \alpha + \beta\xi^l + \gamma\xi^{2l} - ql/3$, $\xi^3 = 1$ and the form of the resulting evolution equation is

$$\underline{y}\bar{y} = \frac{q\overline{T_2\bar{\bar{\bar{\lambda}}}}y + a_1\lambda^2\overline{\lambda\overline{\lambda\bar{\bar{\lambda}}}}\nu}{T_2\lambda + qd_1\lambda\overline{\lambda\overline{\lambda\bar{\bar{\lambda}}}}\nu y} \tag{4.22}$$

where we have made the substitution $y = \bar{\bar{\bar{x}}}/\bar{x}$.

In this case, the Lax pair takes the form

$$v(l+1,n) = L(l,\nu(n))v(l,n) \tag{4.23a}$$

$$v(l,n+1) = N(l,\nu(n))v(l,n) \tag{4.23b}$$

where L as before is given by

$$L = \begin{pmatrix} \frac{\bar{x}}{x} & -\frac{\lambda}{\nu x} \\ \frac{-\lambda\bar{x}}{\nu} & 1 \end{pmatrix}, \tag{4.24}$$

and we write N as

$$N = N_2\nu^2 + N_1\nu + N_0 + \frac{N_{-1}}{\nu} + \frac{N_{-2}}{\nu^2} \tag{4.25}$$

where

$$N_2 = \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}$$

$$N_1 = \begin{pmatrix} 0 & -\frac{T_2}{\lambda\bar{\lambda}\bar{\bar{\lambda}}\sigma\bar{\bar{\bar{x}}}} - \frac{a_1}{\bar{\bar{\bar{x}}}}\left(\lambda\frac{\bar{\bar{\bar{x}}}}{\bar{x}} + \bar{\lambda}\frac{x\bar{\bar{\bar{x}}}}{\bar{x}\bar{\bar{x}}} + \bar{\bar{\lambda}}\frac{x}{\bar{\bar{\bar{x}}}}\right) \\ -\frac{\overline{T_2\bar{\bar{x}}}}{\lambda\bar{\lambda}\bar{\bar{\lambda}}\sigma} - d_1\bar{\bar{\bar{x}}}\left(\lambda\frac{\bar{x}}{\bar{\bar{\bar{x}}}} + \bar{\lambda}\frac{\bar{x}\bar{\bar{x}}}{x\bar{\bar{\bar{x}}}} + \bar{\bar{\lambda}}\frac{\bar{\bar{x}}}{x}\right) & 0 \end{pmatrix}$$

$$N_0 = \begin{pmatrix} \frac{T_2\bar{x}}{\sigma\bar{\bar{\bar{x}}}}\left(\frac{\bar{\bar{x}}}{\bar{\bar{\lambda}}\lambda x} + \frac{\bar{\bar{\bar{x}}}\bar{\bar{x}}}{\lambda\bar{\lambda}x\bar{x}} + \frac{1}{\bar{\bar{\lambda}}\lambda}\right) + a_1\left(\lambda\bar{\lambda}\frac{x}{\bar{\bar{\bar{x}}}} + \lambda\bar{\bar{\lambda}}\frac{x\bar{x}}{\bar{\bar{\bar{x}}}\bar{\bar{x}}} + \overline{\lambda\bar{\lambda}}\frac{\bar{x}}{\bar{\bar{\bar{x}}}}\right) & 0 \\ 0 & \frac{\overline{T_2\bar{\bar{x}}}}{\sigma\bar{x}}\left(\frac{x}{\lambda\bar{\bar{\lambda}}} + \frac{x\bar{x}}{\lambda\bar{\lambda}x\bar{\bar{\bar{x}}}} + \frac{1}{\lambda\bar{\bar{\lambda}}}\right) + d_1\left(\lambda\bar{\lambda}\frac{\bar{\bar{x}}}{x} + \lambda\bar{\bar{\lambda}}\frac{\bar{\bar{\bar{x}}}\bar{\bar{x}}}{x\bar{x}} + \overline{\lambda\bar{\lambda}}\frac{\bar{\bar{\bar{x}}}}{\bar{x}}\right) \end{pmatrix}$$

$$N_{-1} = \begin{pmatrix} 0 & -\frac{T_2\bar{\bar{x}}}{\sigma x}\left(\frac{1}{\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}} + \frac{1}{\bar{\lambda}\bar{x}} + \frac{x}{\lambda\bar{x}\bar{\bar{x}}}\right) - a_1\frac{\lambda\bar{\lambda}\bar{\bar{\lambda}}}{\bar{\bar{\bar{x}}}} \\ -\frac{\overline{T_2 x}}{\sigma\bar{\bar{\bar{x}}}}\left(\frac{\bar{\bar{\bar{x}}}}{\bar{\bar{\lambda}}} + \frac{\bar{x}}{\bar{\lambda}} + \frac{\bar{x}\bar{\bar{x}}}{\lambda x}\right) - d_1\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{x}}} & 0 \end{pmatrix}$$

and

$$N_{-2} = \begin{pmatrix} \frac{T_2}{\sigma} & 0 \\ 0 & \frac{\overline{T_2}}{\sigma} \end{pmatrix}.$$

5. Conclusion

In this paper, we have presented a new Lax pair for a lattice, non-autonomous, modified Korteweg–de Vries equation and shown that it forms a consistent multi-dimensional system when considered together with its Lax pair. We also gave reductions of this non-autonomous

LMKdV to q-difference Painlevé equations and found the Lax pairs corresponding to those reduced equations. A notable feature of these results is that they provide 2×2 Lax pairs for the first time for these versions of qP_{II}, qP_{III} and qP_{V}.

It is worth noting that only one simple form of reduction was investigated here. It remains to be seen whether other types of reductions, that is other forms of $f(\mathbf{x})$ in (3.1), can be used with the LMKdV or other lattice equations.

We note that, so far, there appears not to be a direct method for reducing the lattice Lax pair L, M to the ordinary difference equation's Lax pair L, N. There is a jump in our process of finding N after reduction. The main obstacle is that it is not known whether equations of the form (4.3) can be solved to find N directly. Instead, we have chosen to use the form of the equation to motivate the dependence of N on the spectral parameter ν and then used the compatibility conditions to deduce the entries of N.

A feature of the Lax pairs we deduce is that they share the same L. We note here that the calculation of a series of Lax pairs is also possible for other reductions, including the higher-order difference equations found in section 2. These would also share the same L matrix. This is analogous to the case of integrable differential-equation hierarchies and suggests the existence of a hierarchy for each of the reductions we have studied here. An open problem is to find reductions of lattice equations to infinite hierarchies of q-difference equations along with their Lax pairs. It would be eventually interesting to find reductions from lattice equations to the q-Garnier hierarchy constructed by Sakai in [23].

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